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Electrostatics and magnetostatics in the Schwarzschild metric

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Abstract. The electrostatic field of a point charge at rest in the Schwarzschild metric is given in algebraic form using some results of Copson. It is possible also to determine the magnetostatics. As an example, the magnetostatic field of a current loop surrounding a black hole is given in integral form.

1. Introduction

The generation of an electromagnetic field by static sources in the Schwarzschild background has been considered recently in several papers. Cohen and Wald (1971) and Hanni and Ruffini (1973) have discussed the electric field of a point charge. Petterson (1974) has discussed the magnetic field of a current loop surrounding a Schwarzschild black hole. These solutions are given in the form of series of multipoles.

However, the potential of a point charge at rest in the Schwarzschild metric outside the horizon has been given in algebraic form by Copson (1928); yet a correction of Copson's solution will be given in this paper.

Firstly, we shall examine Copson's solution originally given in isotropic coordinates. Applying Gauss's law, we shall derive the necessary correction. In order to extend the corrected solution of Copson inside the horizon, we shall use the standard Schwarzschild coordinates. We shall see that it is regular everywhere except at the position of the charge.

Secondly, we shall show that it is possible to determine the magnetostatics in the Schwarzschild metric. As an example, we shall determine the magnetic field of a current loop surrounding a black hole.

2. Maxwell's equations in the Schwarzschild metric

The electromagnetic field is assumed to be sufficiently weak that its gravitational effect is negligible. The Einstein–Maxwell equations thus reduce to Maxwell's equations in a Schwarzschild background.

The Schwarzschild metric, in standard coordinates, is:

$$ds^2=\left(1-\frac{2m}{r}\right)dt^2-\left(1-\frac{2m}{r}\right)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2). \qquad (1)$$

Copson uses isotropic coordinates defined by the following transformation formula:

$$\bar{r} = \frac{r - m + [r(r-2m)]^{1/2}}{2} \qquad r > 2m. \tag{2}$$

The inverse formula is:

$$r = \bar{r}\left(1 + \frac{m}{2\bar{r}}\right)^2 \qquad \bar{r} > m/2. \tag{3}$$

In isotropic coordinates, the metric (1) takes the form:

$$ds^2 = \frac{[1-(m/2\bar{r})]^2}{[1+(m/2\bar{r})]^2}\, dt^2 - \left(1 + \frac{m}{2\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2\, d\theta^2 + \bar{r}^2 \sin^2\theta\, d\phi^2). \tag{4}$$

It is known that the isotropic coordinates do not cover the whole Schwarzschild manifold, but only the part lying outside the horizon. From (2), we see that the horizon, $r = 2m$, corresponds to $\bar{r} = m/2$.

Maxwell's equations have the general form:

$$\partial_\rho(\sqrt{-g}F^{\rho\mu}) = \sqrt{-g}J^\mu \qquad \text{and} \qquad \partial_\rho F_{\lambda\mu} + \partial_\lambda F_{\mu\rho} + \partial_\mu F_{\rho\lambda} = 0 \tag{5}$$

where g is the determinant of the metric and J^μ is the current density.

We shall consider Maxwell's equations in the standard coordinates (1). We introduce the following notations:

$$E_i = F_{0i} \qquad \text{and} \qquad B_i = F_{kl} \qquad (i, k, l \text{ circular permutation}). \tag{6}$$

For electrostatics, characterized by $B_i = 0$ and $\partial E_i/\partial t = 0$, from (5) we have:

$$\begin{aligned} &\frac{\partial}{\partial\theta}E_\phi - \frac{\partial}{\partial\phi}E_\theta = 0 \\ &\frac{\partial}{\partial\phi}E_r - \frac{\partial}{\partial r}E_\phi = 0 \\ &\frac{\partial}{\partial r}E_\theta - \frac{\partial}{\partial\theta}E_r = 0 \end{aligned} \tag{7}$$

$$\frac{\partial}{\partial r}\left(\sin\theta\, r^2 E_r\right) + \frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{1-(2m/r)}E_\theta\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{1}{1-(2m/r)}E_\phi\right) = r^2 \sin\theta J^0.$$

We shall use the electrostatic potential V to which the vector potential A_μ reduces in this case:

$$A_0 = V \qquad \text{and} \qquad A_i = 0. \tag{8}$$

The electric field is given by:

$$E_r = -\partial V/\partial r, \qquad E_\theta = -\partial V/\partial\theta \qquad \text{and} \qquad E_\phi = -\partial V/\partial\phi. \tag{9}$$

Combining with (7), we obtain the following equation for the electrostatic potential:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{1-(2m/r)}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{[1-(2m/r)]\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = -r^2 J^0. \tag{10}$$

For magnetostatics, characterized by $E_i = 0$ and $\partial B_i/\partial t = 0$, from (5) we have:

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\left[\left(1-\frac{2m}{r}\right)\sin\theta B_\phi\right]-\frac{\partial}{\partial\phi}\left[\left(1-\frac{2m}{r}\right)\frac{1}{\sin\theta}B_\theta\right]=r^2\sin\theta J^r\\
&\frac{\partial}{\partial\phi}\left(\frac{1}{r^2\sin\theta}B_r\right)-\frac{\partial}{\partial r}\left[\sin\theta\left(1-\frac{2m}{r}\right)B_\phi\right]=r^2\sin\theta J^\theta\\
&\frac{\partial}{\partial r}\left[\left(1-\frac{2m}{r}\right)\frac{1}{\sin\theta}B_\theta\right]-\frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}B_r\right)=r^2\sin\theta J^\phi\\
&\frac{\partial}{\partial r}B_r+\frac{\partial}{\partial\theta}B_\theta+\frac{\partial}{\partial\phi}B_\phi=0.
\end{aligned}\tag{11}$$

Outside the current which is the source of the magnetostatic field, it is usual to introduce the magnetostatic potential Ω. The magnetic field is given by:

$$B_r=-r^2\sin\theta\frac{\partial\Omega}{\partial r},\qquad B_\theta=-\frac{\sin\theta}{1-(2m/r)}\frac{\partial\Omega}{\partial\theta}\qquad\text{and}\qquad B_\phi=-\frac{1}{[1-(2m/r)]\sin\theta}\frac{\partial\Omega}{\partial\phi}.\tag{12}$$

3. Electrostatics

3.1. The solution of Copson

Copson (1928) employs a radial coordinate equal to the $\bar{r}$ defined by (2) divided by $m/2$. Here, we use the isotropic coordinates directly.

In the isotropic coordinates, the electrostatic potential V satisfies the partial differential equation:

$$\Delta V+\frac{(1-(m/2\bar{r})}{[1+(m/2\bar{r})]^3}\frac{\partial}{\partial\bar{r}}\left(\frac{[1+(m/2\bar{r})]^3}{1-(m/2\bar{r})}\right)\frac{\partial}{\partial\bar{r}}V=-\left(1+\frac{m}{2\bar{r}}\right)^2\left(1-\frac{m}{2\bar{r}}\right)^2J^0\tag{13}$$

where Δ is Laplace's operator.

Copson obtains a solution V_C of equation (13) corresponding to a charge e situated at a point $\bar{r}=\bar{a}$ and $\theta=0$ with $\bar{a}>m/2$. This solution is:

$$V_C(\bar{r},\theta)=\frac{e}{[1+(m/2\bar{a})]^2\bar{r}[1+(m/2\bar{r})]^2}\left(\mu(\bar{r},\theta)+\frac{\bar{a}_1}{\bar{a}}\frac{1}{\mu(\bar{r},\theta)}\right)\tag{14}$$

where $\mu(\bar{r},\theta)$ has the following expression:

$$\mu(\bar{r},\theta)=\left(\frac{\bar{r}^2+\bar{a}_1^2-2\bar{a}_1\bar{r}\cos\theta}{\bar{r}^2+\bar{a}^2-2\bar{a}\bar{r}\cos\theta}\right)^{1/2}\tag{15}$$

with $\bar{a}_1=m^2/4\bar{a}$.

However, it is easy to see that the solution (14) has also some other source, besides the charge e at point $(\bar{a},0,0)$. Indeed, for $\bar{r}\to\infty$, the electrostatic potential V_C has the asymptotic form:

$$V_C(\bar{r},\theta)\sim\left(1-\frac{m}{\bar{a}[1+(m/2\bar{a})]^2}\right)\frac{e}{\bar{r}}.\tag{16}$$

Consequently by virtue of Gauss's theorem, it also contains a second charge equal to

$$-em/\bar{a}[1+(m/2\bar{a})]^2.$$

Since Copson's solution has only the source e outside the horizon, this second source must lie inside the horizon. It is possible to see that the electric field corresponding to electrostatic potential V_C is regular at the horizon. Now, the only electric field which is regular for $m/2 \leq \bar{r} < \infty$ is spherically symmetric (Israel 1968) and consequently, it will be of the form:

$$\frac{\text{constant}}{\bar{r}[1+(m/2\bar{r})]^2}. \tag{17}$$

The value of the constant corresponds to the second charge at $\bar{r} \to \infty$:

$$\text{constant} = \frac{em}{\bar{a}[1+(m/2\bar{a})]^2}.$$

It follows that the electrostatic potential which has as source the charge e at $(\bar{a}, 0, 0)$ is:

$$V(\bar{r}, \theta) = V_C(\bar{r}, \theta) + \frac{em}{\bar{a}[1+(m/2\bar{a})]^2}\,\frac{1}{\bar{r}[1+(m/2\bar{r})]^2}. \tag{18}$$

3.2. *Electrostatic potential*

Cohen and Wald (1971) and also Hanni and Ruffini (1973) derive the electrostatic potential V using a multipole expansion in standard coordinates. We remark that this multipole expansion coincides with the one given by Copson (1928), when the correction given in (18) has been taken into account.

In order to determine the field also inside the horizon, we shall use the standard coordinates.

So as to express $V(\bar{r}, \theta)$ given by (18) in standard coordinates, we write it in the following form:

$$V(\bar{r}, \theta) = \frac{e}{[1+(m/2\bar{a})]^2\bar{r}[1+(m/2\bar{r})]^2}\left(\mu^2 + \frac{\bar{a}_1^2}{\bar{a}^2}\frac{1}{\mu^2} + \frac{2\bar{a}_1}{\bar{a}}\right)^{1/2} + \frac{em}{\bar{a}[1+(m/2\bar{a})]^2}\,\frac{1}{\bar{r}[1+(m/2\bar{r})]^2}. \tag{19}$$

From the transformation formula (3), we deduce the following relations:

$$\begin{aligned} r - m &= \bar{r} + \frac{\bar{a}\bar{a}_1}{\bar{r}} \\ (r-m)^2 - \frac{m^2}{2} &= \bar{r}^2 + \frac{\bar{a}^2\bar{a}_1^2}{\bar{r}^2}. \end{aligned} \tag{20}$$

We also define the quantity a corresponding to $\bar{a}$ by the transformation formula (3):

$$a = \bar{a}\left(1+\frac{m}{2\bar{a}}\right)^2, \qquad a > 2m. \tag{21}$$

A straightforward calculation gives:

$$V(r,\theta)=\frac{e}{ar}\frac{(r-m)(a-m)-m^2\cos\theta}{[(r-m)^2+(a-m)^2-m^2-2(r-m)(a-m)\cos\theta+m^2\cos^2\theta]^{1/2}}+\frac{em}{ar}. \tag{22}$$

It is easy to see that $V(r, \theta)$ is well behaved everywhere except at the position of the charge e.

The formula giving the electrostatic potential $V(r\theta\phi, a\theta_0\phi_0)$ of a charge e situated at the point (a, θ_0, ϕ_0) with $a>2m$ is obtained from (22) by simply replacing $\cos\theta$ by $\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)$. We write the result since it will be needed in the next section:

$$V(r\theta\phi, a\theta_0\phi_0)$$
$$=\frac{e}{ar}\frac{(r-m)(a-m)-m^2\lambda(\theta,\phi)}{[(r-m)^2+(a-m)^2-m^2-2(r-m)(a-m)\lambda(\theta,\phi)+m^2\lambda^2(\theta,\phi)]^{1/2}}+\frac{em}{ar} \tag{23}$$

with:

$$\lambda(\theta,\phi)=\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0).$$

4. Magnetostatics

4.1. Method

Outside the electromagnetic sources, it is possible by duality to transform an electrostatic solution into a magnetostatic one and vice versa. But the solution corresponding to a given source is not obvious.

Using the equations (7), we see that the component of the electrostatic field E_r satisfies the partial differential equation:

$$\frac{\partial}{\partial r}\left[\left(1-\frac{2m}{r}\right)\frac{\partial}{\partial r}r^2E_r\right]+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}E_r\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\rho^2}E_r=\frac{\partial}{\partial r}\left[\left(1-\frac{2m}{r}\right)r^2J^0\right]. \tag{24}$$

Using also the equation (11) for the magnetostatic field, if we introduce the function X defined by:

$$X=\frac{1}{r^2\sin\theta}B_r \tag{25}$$

we see that the function X satisfies the partial differential equation:

$$\frac{\partial}{\partial r}\left[\left(1-\frac{2m}{r}\right)\frac{\partial}{\partial r}r^2X\right]+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}X\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}X$$
$$=-\frac{r^2}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta J^\phi\right)+\frac{r^2}{\sin\theta}\frac{\partial}{\partial\phi}J^\theta. \tag{26}$$

The equations (24) and (26) without the source term are identical, in agreement with duality.

We shall determine the solution of the equation (26) for the source term:

$$4\pi\delta(r-r_0)\delta(\cos\theta-\cos\theta_0)\delta(\phi-\phi_0). \tag{27}$$

This solution will then allow us to determine general magnetostatic fields in the Schwarzschild background. According to (12) and (25), the corresponding magnetostatic potential $\Omega(r\theta\phi, r_0\theta_0\phi_0)$ will satisfy the equation:

$$X=-\partial\Omega/\partial r. \tag{28}$$

In order to obtain the source term (27) in the equation (24), we must consider the following charge density:

$$J^0(r,\theta)=\frac{4\pi}{r^2[1-(2m/r)]}Y(r-r_0)\delta(\cos\theta-\cos\theta_0)\delta(\phi-\phi_0) \tag{29}$$

$Y(r-r_0)$ being the step function.

Remembering the first equation (9), we see that, for the same source term in equation (24) and (26), we shall have $\Omega=V$. The solution $V=\Omega(r\theta\phi, r_0\theta_0\phi_0)$ of equation (24) for the source term (27) is obtained from (23) with the help of the remark that the charge density corresponding to $V(r\theta\phi, a\theta_0\phi_0)$ is:

$$J^0(r,\theta)=\frac{4\pi}{r^2}\delta(r-a)\delta(\cos\theta-\cos\theta_0)\delta(\phi-\phi_0). \tag{30}$$

We thus obtain:

$$\Omega(r\theta\phi, r_0\theta_0\phi_0)=\int_{r_0}^{\infty}\frac{1}{1-(2m/a)}V(r\theta\phi, a\theta_0\phi_0)\,\mathrm{d}a. \tag{31}$$

It is easy to show that this solution is well behaved everywhere except at (r_0, θ_0, ϕ_0).

4.2. Current loop at radius r_0

We consider a static current loop at radius r_0, with $r_0>2m$, located in the plane $\theta=\pi/2$. The only non-vanishing component of the current density is now:

$$J^\phi(r,\theta)=4\pi\frac{J}{r^2}\delta(r-r_0)\delta(\cos\theta). \tag{32}$$

However, J^ϕ is a current density. Consequently using the total current I, we find that J has the following expression (Petterson 1974):

$$J=\left(1-\frac{2m}{r_0}\right)^{1/2}I. \tag{33}$$

For the current density (32), the source term of the equation (26) is:

$$\frac{\mathrm{d}}{\mathrm{d}\cos\theta}\delta(\cos\theta)\delta(r-r_0) \quad \text{or} \quad -\frac{\mathrm{d}}{\mathrm{d}\cos\theta_0}\delta(\cos\theta-\cos\theta_0)|_{\theta_0=\pi/2}\delta(r-r_0). \tag{34}$$

Comparing (34) with (27), we see that the magnetostatic potential corresponding to the source term (34) will be obtained from (31) by the formula:

$$\Omega(r,\theta)=-J\left(\int_0^{2\pi}\frac{\mathrm{d}}{\mathrm{d}\cos\theta_0}\Omega(r\theta\phi, r_0\theta_0\phi_0)\,\mathrm{d}\phi_0\right)_{\theta_0=\pi/2}. \tag{35}$$

The quantity (35) can be written explicitly in integral form. We obtain:

$$\Omega(r, \theta) = -J \int_0^{2\pi} \int_{r_0}^{\infty} \frac{a(r-2m)\cos\theta \, da \, d\phi_0}{[(r-m)^2 + (a-m)^2 - m^2 - 2(r-m)(a-m)\sin\theta\cos(\phi-\phi_0) + m^2\sin^2\theta\cos^2(\phi-\phi_0)]^{3/2}}. \tag{36}$$

The magnetostatic potential is antisymmetric. The angle ϕ can be eliminated and the final result is:

$$\Omega(r, \theta) = -J(r-2m)\cos\theta \times \int_0^{2\pi} \int_{r_0}^{\infty} \frac{a \, da \, d\psi}{[(r-m)^2 + (a-m)^2 - m^2 - 2(r-m)(a-m)\sin\theta\cos\psi + m^2\sin^2\theta\cos^2\psi]^{3/2}}. \tag{37}$$

We see that the integral (37) is finite when $r_0 \to 2m$. But then we conclude from the relation (33) that for any given intensity I, the quantity $\Omega(r, \theta)$, and consequently the magnetic field, vanishes as $r_0 \to 2m$. This property has been proved by Petterson (1974), but only for the dipole part of the magnetic field.

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